Derivative of a Composite Function

EXAMPLE 1 Relating Derivatives

outside

The function $y = 6x - 10 = \boxed{2(3x - 5)}$ is the composite of the functions $y = 2u$ and $u = 3x - 5$. How are the derivatives of these three functions related?

inside

$$\frac{dy}{dx} = 6 \qquad \frac{dy}{du} = 2 \qquad \frac{du}{dx} = 3$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

derivative of outside

derivative of the inside

EXAMPLE 2 Relating Derivatives

outside

The polynomial $y = 9x^4 + 6x^2 + 1 = (3x^2 + 1)^2$ is the composite of $y = u^2$ and $u = 3x^2 + 1$. Calculating derivatives, we see that

inside

$$\frac{dy}{dx} = 36x^3 + 12x \qquad \frac{dy}{du} = 2u \qquad \frac{du}{dx} = 6x$$

$2(3x^2+1)$

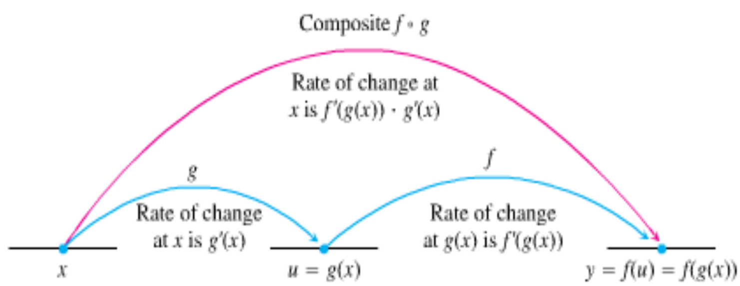
$6x^2+2$

f(x) = u^2 g(x) = $3x^2+1$ f '(x) = $2u$ g'(x) = $6x$

$(3x^2+1)^2$ $2(3x^2+1)$

Derivative of the outside in terms of the inside times the derivative of the inside

$f'(g(x)) \cdot g'(x)$



RULE 8 The Chain Rule

If f is differentiable at the point $u = g(x)$, and g is differentiable at x, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x, and

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx},$$

where dy/du is evaluated at $u = g(x)$.

EXAMPLE 3 Applying the Chain Rule

An object moves along the x-axis so that its position at any time $t \geq 0$ is given by $x(t) = \cos(t^2 + 1)$. Find the velocity of the object as a function of t.

outside $\rightarrow$ $\cos u$ $(f(x))$

inside $\rightarrow$ $t^2 + 1$ $(g(x))$

$v(t) = -\sin(t^2+1) \cdot 2t$

$= -2t\sin(t^2+1)$

"Outside-Inside" Rule

It sometimes helps to think about the Chain Rule this way: If $y = f(g(x))$, then

$$\frac{dy}{dx} = f'(g(x)) \cdot g'(x).$$

In words, differentiate the "outside" function f and evaluate it at the "inside" function $g(x)$ left alone; then multiply by the derivative of the "inside function."

EXAMPLE 4 Differentiating from the Outside in

Differentiate $\sin(x^2 + x)$ with respect to x.

SOLUTION

$$\frac{d}{dx}\sin\underbrace{(x^2 + x)}_{\text{inside}} = \cos\underbrace{(x^2 + x)}_{\substack{\text{inside}\\ \text{left alone}}} \cdot \underbrace{(2x + 1)}_{\substack{\text{derivative of}\\ \text{the inside}}}$$

Repeated Use of the Chain Rule

EXAMPLE 5 **A Three-Link "Chain"**

Find the derivative of $g(t) = \tan(5 - \sin 2t)$.

$\tan u \to$ outside

$5 - \sin 2t \to$ inside

$$g'(t) = \sec^2(5 - \sin 2t) \cdot \underbrace{-\cos(2t) \cdot 2}$$

$$= -2\sec^2(5 - \sin(2t))\cos(2t)$$

Power Chain Rule

If f is a differentiable function of u, and u is a differentiable function of x, then substituting $y = f(u)$ into the Chain Rule formula

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

leads to the formula

$$\frac{d}{dx} f(u) = f'(u)\frac{du}{dx}.$$

Here's an example of how it works: If n is an integer and $f(u) = u^n$, the Power Rules (Rules 2 and 7) tell us that $f'(u) = nu^{n-1}$. If u is a differentiable function of x, then we can use the Chain Rule to extend this to the **Power Chain Rule:**

$$\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx}. \qquad \frac{d}{du}(u^n) = nu^{n-1}$$

EXAMPLE 7 Finding Slope

(a) Find the slope of the line tangent to the curve $y = \sin^5 x$ at the point where $x = \pi/3$.

(b) Show that the slope of every line tangent to the curve $y = 1/(1 - 2x)^3$ is positive.

a) $y = (\sin x)^5$

outside → u^5

inside → $\sin x$

$$y' = 5(\sin x)^4 \cdot \cos x$$

$$= 5\left(\frac{\sqrt{3}}{2}\right)^4 \cdot \frac{1}{2}$$

$$= \frac{5}{2} \cdot \frac{9}{16} = \frac{45}{32}$$

b) $1/(1-2x)^3$

$(1-2x)^{-3}$

$$y' = -3(1-2x)^{-4} \cdot -2$$

$$= \frac{6}{(1-2x)^4}$$

6 → always +

$(1-2x)^4$ → even power ∴ always +

EXAMPLE 8 Radians Versus Degrees

It is important to remember that the formulas for the derivatives of both sin x and cos x were obtained under the assumption that x is measured in radians, *not* degrees.